Spatial Doppler anomaly in an excitable medium

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We consider a two-dimensional steady spiral wave in a singly diffusive FitzHugh-Nagumo medium. When perturbed by a uniform external field, the spiral will, in general, display a drifting motion as well as a spatial deformation with Doppler-like features. The present work demonstrates the existence of a radical departure from the conventional moving-source Doppler deformation of the wave pattern in space, even to first order in the perturbation. In particular, the maximal shrinking of the wavelength can occur in a direction very different from the drift itself; the anomalous direction of the shift amounts to a *zeroth-order effect* in the perturbation. We present a simple renormalization-based formula for this effect, as well as the results of some numerical simulations; theory and simulations are in good agreement. The formula involves dispersion properties of the one-dimensional unperturbed system. The basic technique is related to Zykov's derivation of the curvature-speed formula for a wave front. There exists a Doppler anomaly in time as well, but it is less conspicuous than the spatial one because it is of second order in the perturbation. [S1063-651X(96)02608-6]

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I. INTRODUCTION

In this paper we study the spatial Doppler deformation of a drifting spiral wave. More specifically, we consider a snapshot of the spiral, and examine the spacing between its turns in various regions characterized by their orientation with respect to the spiral's center. We assume the medium of propagation to be modeled by two-dimensional reaction-diffusion equations of the FitzHugh-Nagumo type [1]; the drift is caused by an extra gradient term (or convection term) in one of the equations. The Belousov-Zhabotinsky (BZ) reaction [2] in an external uniform time-independent electric field provides a reasonable realization of such a model. That the field has a strong effect on two-dimensional BZ wave patterns, including spirals, is well documented experimentally and in computer simulations [3–7]. However, a quantitative theoretical understanding directly based on the properties of the medium has been lacking, especially in regard to how the deformation differs from traditional expectations. In this work it is our aim to provide such an analysis and to test its predictions through computer simulations.

In the conventional moving-source situation, the wavelength is maximally reduced in the direction of drift. A spiral wave, however, breaks the chiral symmetry of the system; as a consequence, the drift need not be parallel (or antiparallel) to the perturbing field. This is true as well if the field is replaced by a gradient in some parameter of the medium [8]. Furthermore, the direction of maximum Doppler deformation need no longer be parallel to the drift; this is the feature we refer to as being anomalous, and it can be observed in Refs. [4, 7]. Finally, the maximum deformation need not be lined up with the field. These angular differences can be large, and *they do not necessarily vanish in the limit of zero external field*. In what follows we demonstrate these statements numerically and analytically as they apply to the Doppler deformation; in the analytic work, the drift velocity will be assumed known.

The object of this research is to understand and predict, quantitatively, deformations such as are displayed in the simulations of Fig. 1. Our theoretical derivation of the effect results in a formula, Eq. (23) further on, which predicts the deformation on the basis of the spiral's drift velocity and its unperturbed period of rotation; the formula also makes use of some ingredients, such as the dispersion, from the one-dimensional solutions of the unperturbed FitzHugh-Nagumo equations. To first order in the perturbing field, our prediction has the form

$$\Lambda(\mathbf{n})-\Lambda(-\mathbf{n})=\mathbf{n}\cdot(p\mathbf{V}+q\mathbf{G}),$$

where the unit vector **n** points from the spiral's instantaneous center of rotation to the observation region, and where the left side is the wavelength difference between that region and the one oppositely located relative to the center. The vectors **V** and **G** stand for drift velocity and perturbing field; the coefficients p and q involve the above-mentioned one-dimensional parameters. The method of derivation amounts to a kind of renormalization, or rescaling, in which we compare two solutions for plane waves. The first solution is the perturbed spiral wave in the peripheral (spatially asymptotic) region. The second solution is the unperturbed periodic one-dimensional wave, calculated within a range of parameters. No detailed knowledge of the solutions is required in order to derive the formula. The constraint on the Doppler pattern arises from the fact that both solutions obey equations that are formally identical, although some parameters are different. An appropriate rescaling of those parameters must convert one solution into the other. A similar procedure was used some time ago by Zykov [9] in order to estimate the speed of a wave from the front's curvature.

A topic of considerable importance is the Doppler shift in the wave's period rather than in its wavelength. The motivation can arise from the BZ reaction [10] but more typically from electrocardiography [11–18], where the spiral's drift is caused by nonuniformities in the medium, or possibly by spontaneous wandering. The results of the present paper

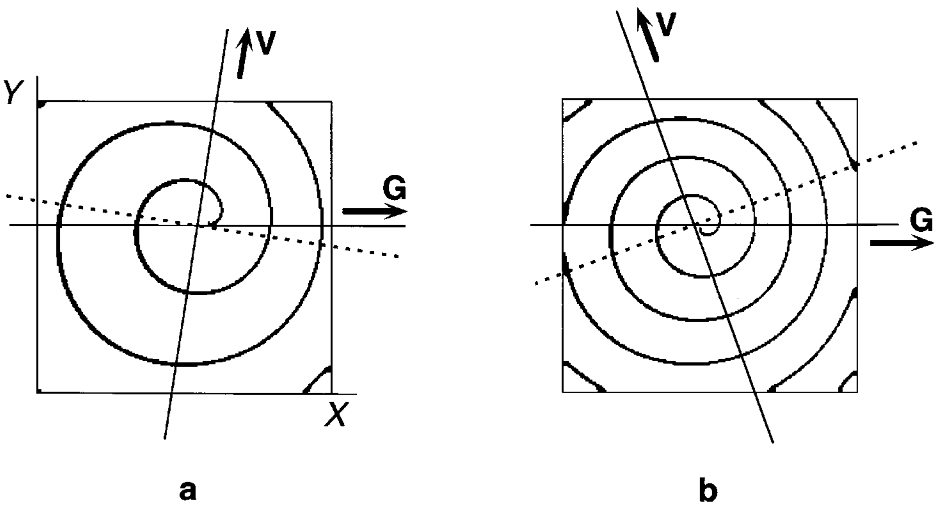

FIG. 1. *Drifting spirals with anomalous Doppler deformations*. Snapshots from two simulations, (a) and (b), based on a FitzHugh-Nagumo medium perturbed by a uniform field **G**; the spiral's drift velocity is **V**. We observe a nonzero Doppler deformation along the dotted line (perpendicularly to **V**), contrary to the conventional expectation. A deformation along **V** is visible as well, clearly so in (a) and marginally in (b). The parameters of the simulations are described in Appendix C; panels (a) and (b) differ in the value of the parameter K_2. The spiral curves in this figure are the loci of maxima for the variable u defined by Eqs. (1) and (2).

have a bearing on these observations if the anomaly we describe is somehow generic, in excitable media, with respect to the detailed origin of the drift. In fact, a temporal anomaly is visible in Ref. [12]. Nevertheless, it must be pointed out that in the time domain the anomalous shift and the normal one differ only *in the second order of perturbation*. Thus, under most experimental conditions, the temporal anomaly will be far less observable than the spatial one. This second-order property is demonstrated in Appendix B. Apart from attempting to clarify that issue, we do not pursue the discussion of the temporal Doppler shift any further.

II. ASYMPTOTIC BEHAVIOR IN THE COMOVING SYSTEM

We study drifting spiral waves in a perturbed FitzHugh-Nagumo medium characterized, in two space dimensions (x,y), by the following equations [1] for two propagating variables u,v:

$$\partial_t u-\alpha\nabla^2 u+G\partial_x u+\Phi_1(u,v)=0, \tag{1}$$

$$\partial_t v+\eta\Phi_2(u,v)=0, \tag{2}$$

where α, G, and η are constant parameters; α is the diffusivity, and G parametrizes a gradient perturbation in the x direction. The parameter η is not necessarily small, and its role will become apparent further on; Φ_1 and Φ_2 are generic reactivity functions, capable of maintaining a spiral wave. In Eq. (1), the operator $\partial_t+G\partial_x$ is the convective time derivative associated with a dimensionless effective ion concentration u in the BZ medium; G is proportional to the electric field and the ion mobility. The spatial domain is considered unbounded.

We postulate that the perturbed spiral, when observed many wavelengths away from its core, drifts at a constant velocity **V**. The core may still do some limited wandering with respect to that uniform motion [19]. Changing from the laboratory coordinates $(x,y)=\mathbf{r}$ to the drifting spiral's comoving coordinates $(X,Y)=\mathbf{R}$,

$$\mathbf{R}=\mathbf{r}-\mathbf{V}t, \tag{3}$$

we have for (1) and (2)

$$\partial_t u-(\mathbf{V}+\mathbf{G})\cdot\nabla u-\alpha\nabla^2 u+\Phi_1(u,v)=0, \tag{4}$$

$$\partial_t v-\mathbf{V}\cdot\nabla v+\eta\Phi_2(u,v)=0, \tag{5}$$

where ∇ now operates with respect to **R**, and where we define $\mathbf{G}=(G,0)$. In the comoving system we postulate a steady-state rotation, i.e., the frequency is constant in time and independent of the (comoving) location. The core region may or may not behave in that manner, but is unimportant for our purpose.

We next examine a peripheral region ($R\to\infty$) in which the wave can be considered plane, so that it propagates in the direction of a fixed unit vector **n**; asymptotically, **n** is in general not parallel to **R** except in the unperturbed case. In such a region we have

$$u=u[\mathbf{n}\cdot\mathbf{R}-C(\mathbf{n})t],\quad v=v[\mathbf{n}\cdot\mathbf{R}-C(\mathbf{n})t], \tag{6}$$

where $C(\mathbf{n})$ is the speed of propagation, in direction **n**, of a plane wave in the perturbed medium. We note once more that we are in the comoving reference frame. Equations (4) and (5) now read

$$-[C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})]u'-\alpha u''+\Phi_1(u,v)=0, \tag{7}$$

$$-[C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}]v'+\eta\Phi_2(u,v)=0, \tag{8}$$

where the prime indicates total differentiation. It is readily verified that Eqs. (6)–(8) are valid in the case of a weakly deformed spiral; specifically, we must assume $|(dC/d\Theta)/C|\ll 1$, where Θ is the polar angle of **n**. If **V** and $(dC/d\Theta)/C$ are both of order G for a small perturbation G, then (7) and (8) are valid through order G. This is the accuracy sought in the present paper.

Equations (7) and (8) are to be compared with the plane-wave equations for unspecified fixed speed c and propagating variables $\mathcal{U}$, $\mathcal{V}$ in the unperturbed medium at rest,

$$-c\mathcal{U}'-\alpha\mathcal{U}''+\Phi_1(\mathcal{U},\mathcal{V})=0, \tag{9}$$

$$-c\mathcal{V}'+\eta\Phi_2(\mathcal{U},\mathcal{V})=0. \tag{10}$$

We assume that $\mathcal{U}$ and $\mathcal{V}$ are periodic functions with the same period. The total derivatives in (9) and (10) are defined with respect to $X-ct$, where X is the one-dimensional space coordinate.

III. PLANE-WAVE RENORMALIZATION

The system (7), (8) can be recast into the standard form (9), (10). Specifically, we leave (7) intact, but multiply (8) by a constant factor:

$$-[C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})]v' + \frac{C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})}{C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}}\,\eta\Phi_2(u,v) = 0. \tag{11}$$

Comparing the two systems, we see that (7), (11) is just a rescaling of (9), (10) under

$$c \rightarrow C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G}), \tag{12}$$

$$\eta \rightarrow \frac{C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})}{C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}}\,\eta. \tag{13}$$

We are interested in solving for the cycle wavelength λ as a function of direction. In the unperturbed case (9), (10), it is known [20] that under periodic boundary conditions, and assuming a stable solution, λ in a plane wave is completely determined by c, α, η, and the detailed form of Φ_1 and Φ_2.

Let α be permanently fixed, and let the functional form of Φ_1 and Φ_2 be similarly fixed. Then, in a one-dimensional periodic wave, λ may be viewed as a function of c and η:

$$\lambda = \lambda(c,\eta). \tag{14}$$

As $R\rightarrow\infty$, let the drifting spiral have a cycle wavelength $\Lambda(\mathbf{n})$. Then, according to (12) and (13), we have a "renormalization condition"

$$\Lambda(\mathbf{n}) = \lambda\left(C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G}),\ \frac{C(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})}{C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}}\,\eta\right), \tag{15}$$

where λ is the same function as in (14). That function is known, in principle, if the one-dimensional problem has been solved in a range of c and η. We now convert (15) into a constraint for C by eliminating Λ in favor of C:

$$\Lambda(\mathbf{n}) = \frac{C(\mathbf{n})}{F}, \tag{16}$$

where F is the perturbed frequency, independent of **n**. The isotropy of F is an essential ingredient in this argument; it means that **V** has been chosen correctly.

IV. FIRST-ORDER PERTURBATION FORMULAS

We examine the perturbative case

$$C(\mathbf{n}) = c_0 + c_1(\mathbf{n}), \tag{17}$$

$$\Lambda(\mathbf{n}) = \lambda_0 + \lambda_1(\mathbf{n}), \tag{18}$$

$$F = f_0 + f_1, \tag{19}$$

where c_1, λ_1, and f_1 are first order in G, and where c_0, λ_0, and f_0 apply to the peripheral region of the unperturbed spiral for given η. We also assume **V** to be first order in G.

Equation (15), expanded to first order in G and with (16) for the left side, becomes

$$\lambda_0\left[1+\frac{c_1(\mathbf{n})}{c_0}-\frac{f_1}{f_0}\right] = \lambda\left(c_0+c_1(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G}),\ \left(1+\frac{\mathbf{n}\cdot\mathbf{G}}{c_0}\right)\eta\right), \tag{20}$$

or, with the zeroth order subtracted,

$$\lambda_0\left[\frac{c_1(\mathbf{n})}{c_0}-\frac{f_1}{f_0}\right] = [c_1(\mathbf{n})+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})]\left(\frac{\partial\lambda}{\partial c}\right)_\eta + \frac{\mathbf{n}\cdot\mathbf{G}}{c_0}\,\eta\left(\frac{\partial\lambda}{\partial\eta}\right)_c, \tag{21}$$

the constant variables being denoted by subscripts, and $c=c_0$ being understood. Solving for c_1, we have

$$c_1(\mathbf{n}) = \left[\frac{\lambda_0 f_1}{f_0}+\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})\left(\frac{\partial\lambda}{\partial c}\right)_\eta + \frac{\mathbf{n}\cdot\mathbf{G}}{c_0}\,\eta\left(\frac{\partial\lambda}{\partial\eta}\right)_c\right] \Bigg/ \left[\frac{\lambda_0}{c_0}-\left(\frac{\partial\lambda}{\partial c}\right)_\eta\right]. \tag{22}$$

In the limit of zero dispersion, $\mathcal{D}\equiv(\partial c/\partial f\,)_\eta\rightarrow 0$, both the numerator and denominator of (22) are infinite. This is related to the fact that our renormalization argument breaks down if $\mathcal{D}=0$ exactly. Indeed, zero dispersion would mean that only one value of c can exist, and Eq. (14) would be meaningless. Fortunately, it is safe to assume [20] that there always exists at least a small finite range of c. Because of this singularity, and for purposes of measurement as well, it can be more convenient to take the frequency or period parameter, f or $\tau=1/f$, as independent. As an illustration we select τ_1, as well as λ_1, $(\partial\lambda/\partial\tau)_\eta$, and $(\partial\tau/\partial\eta)_\lambda$ rather than c_1, $(\partial\lambda/\partial c)_\eta$ and $(\partial\lambda/\partial\eta)_c$, as the ingredients of the formula. The partial-derivative manipulations are shown in Appendix A. The result, equivalent to (22), is

$$\lambda_1(\mathbf{n}) = \frac{\tau_0}{\lambda_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta \left\{\lambda_0\,\frac{\tau_1}{\tau_0}-\mathbf{n}\cdot\left[(\mathbf{V}+\mathbf{G})\tau_0+\mathbf{G}\eta\left(\frac{\partial\tau}{\partial\eta}\right)_\lambda\right]\right\}. \tag{23}$$

We need to comment on several features of this formula, which is our central theoretical result. It yields the excess wavelength λ_1 in direction **n** in terms of the period τ_0 and asymptotic wavelength λ_0 of the unperturbed spiral. The dispersion-like quantities $(\partial\lambda/\partial\tau)_\eta$ and $(\partial\tau/\partial\eta)_\lambda$ are measured from the unperturbed plane waves with period τ_0 (and therefore wavelength λ_0); the parameter η must be assigned the same value for the perturbed spiral, the unperturbed spiral, and the unperturbed plane wave. We can in fact set, without loss of generality, $\eta=1$ in Eq. (2), and here, after the partial derivative has been performed. The perturbing vector **G** is given, and the drift velocity **V** must be measured from the perturbed spiral. The scalar contribution τ_1 (the change in period due to the presence of **G**) is not predicted by our calculations. However, in a snapshot of the drifting spiral, τ_1 is irrelevant to a measurement of $\lambda_1(\mathbf{n})-\lambda_1(-\mathbf{n})$, which gives the Doppler geometry. The interpretation of (23) is facilitated by the fact that **n** and **R** can be considered parallel; any correction must contribute to (23) in higher than first-order perturbation. In practice we may sometimes want to rewrite the last term of (23) by using the identity $(\partial\lambda/\partial\tau)_\eta(\partial\tau/\partial\eta)_\lambda=-(\partial\lambda/\partial\eta)_\tau$.

The term in **V** still represents (essentially) the conventional Doppler shift for a moving source. In the anomalous situation which concerns us here, and in the zero-dispersion limit $D\to0$, the overall factor in (23) becomes $(\tau_0/\lambda_0)(\partial\lambda/\partial\tau)_\eta=1$. If we now measure the Doppler shift at right angles to **G**, Eq. (23) becomes

$$\lambda_1(\mathbf{n})=\lambda_0\left[\frac{\tau_1}{\tau_0}-\mathbf{n}\cdot\frac{\mathbf{V}}{c_0}\right],\quad(\mathcal{D}=0,\quad \mathbf{n}\cdot\mathbf{G}=0),\tag{24}$$

yielding the conventional pattern for $\lambda_1(\mathbf{n})-\lambda_1(-\mathbf{n})$. As far as the terms in **G** are concerned, they are kinematically equivalent to an additional deformation caused by a ''wind'' of velocity $\tau_0+\eta\,(\partial\tau/\partial\eta)_\lambda$ in the $-\mathbf{G}$ direction.

V. NUMERICAL TESTS

In order to test formula (23) we turn to simulations (a) and (b), used to produce Fig. 1. These simulations are described in Appendix C. From the snapshots we measure $\delta\equiv\frac{1}{2}[\Lambda(\mathbf{n})-\Lambda(-\mathbf{n})]$ with **n** in the $+X$ or $+Y$ directions. On the other hand, noting that $\mathbf{G}=(0.16,\,0)$ and $\eta=1$ in both simulations, we predict from (23)

$$\delta_X=-\frac{\tau_0}{\lambda_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta\left[(V_X+0.16)\tau_0+(0.16)\left(\frac{\partial\tau}{\partial\eta}\right)_\lambda\right],\tag{25}$$

$$\delta_Y=-\frac{\tau_0}{\lambda_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta V_Y\tau_0.\tag{26}$$

The values of λ_0 and τ_0 were measured from the unperturbed spiral: $(\lambda_0,\tau_0)=(324,\,36.4)$ for simulation (a), and $(\lambda_0,\tau_0)=(25.6,\,23.2)$ for (b). The partial derivatives were obtained from one-dimensional simulations, in which η and τ were varied; λ_0 can alternatively be obtained from these simulations on the basis of the known τ_0. In this way we find for the overall coefficient $(\tau_0/\lambda_0)(\partial\lambda/\partial\tau)_\eta$ in both cases (a) and (b), the value 1.00 within 5%, indicating low dispersion. From the drifting spirals we find $\mathbf{V}=(0.0374,\,0.130)$ (a) and $\mathbf{V}=(-0.0271,\,0.0651)$ (b). These values were measured by following the core for five rotation cycles in (a) and 12 in (b); **V** was thereby verified to be constant. From Eqs. (25) and (26) we then predict for (δ_X,δ_Y) the values $(-7.7,-4.8)$ (a) and $(-3.2,-1.5)$ (b). As measured directly from the snapshots, which permit about 10% precision, these data are $(-8.0,-4.8)$ (a) and $(-3.2,-1.4)$ (b). We conclude that the first-order theory agrees with the numerical simulation within the latter's accuracy.

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APPENDIX A: CHANGING VARIABLES IN THE FIRST-ORDER FORMULA

To pass from (22) to (23) we use (16)–(19) on the left side; in the denominator on the right side we use

$$\left(\frac{\partial\lambda}{\partial c}\right)_\eta=\left(\frac{\partial}{\partial c}\,(c\,\tau)\right)_\eta=\tau_0+c_0\left(\frac{\partial\tau}{\partial c}\right)_\eta.\tag{A1}$$

Equation (22) becomes

$$\frac{c_0}{\lambda_0}\lambda_1=\left[\frac{c_0}{\tau_0}+\left(\frac{\partial c}{\partial\tau}\right)_\eta\right]\tau_1-\frac{\mathbf{n}\cdot(\mathbf{V}+\mathbf{G})}{c_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta-\frac{\mathbf{n}\cdot\mathbf{G}}{c_0^2}\,\eta\left(\frac{\partial\lambda}{\partial\eta}\right)_c\left(\frac{\partial c}{\partial\tau}\right)_\eta.\tag{A2}$$

The term in τ_1 can be simplified by

$$\left(\frac{\partial c}{\partial\tau}\right)_\eta=\left[\frac{\partial}{\partial\tau}\left(\frac{\lambda}{\tau}\right)\right]_\eta=\frac{1}{\tau_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta-\frac{\lambda_0}{\tau_0^2},\tag{A3}$$

giving

$$\left[\frac{c_0}{\tau_0}+\left(\frac{\partial c}{\partial\tau}\right)_\eta\right]\tau_1=\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta\frac{\tau_1}{\tau_0}.\tag{A4}$$

Next we rewrite the last term of (A2). Considering $\lambda=\lambda(c,\eta)$, we have

$$\left(\frac{\partial\lambda}{\partial\eta}\right)_\tau=\left(\frac{\partial\lambda}{\partial c}\right)_\eta\left(\frac{\partial c}{\partial\eta}\right)_\tau+\left(\frac{\partial\lambda}{\partial\eta}\right)_c\tag{A5}$$

or, multiplying by $(\partial c/\partial\tau)_\eta$ and rearranging terms,

$$\begin{aligned}\left(\frac{\partial\lambda}{\partial\eta}\right)_c\left(\frac{\partial c}{\partial\tau}\right)_\eta&=\left(\frac{\partial\lambda}{\partial\eta}\right)_\tau\left(\frac{\partial c}{\partial\tau}\right)_\eta-\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta\left(\frac{\partial c}{\partial\eta}\right)_\tau\\&=\left(\frac{\partial\lambda}{\partial\eta}\right)_\tau\left[\frac{1}{\tau_0}\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta-\frac{\lambda_0}{\tau_0^2}\right]\\&\quad-\left(\frac{\partial\lambda}{\partial\tau}\right)_\eta\left[\frac{1}{\tau_0}\left(\frac{\partial\lambda}{\partial\eta}\right)_\tau\right]=-\frac{\lambda_0}{\tau_0^2}\left(\frac{\partial\lambda}{\partial\eta}\right)_\tau.\end{aligned}\tag{A6}$$

Substituting (A4) and (A6) in (A2) yields (23).

APPENDIX B: THE FREQUENCY ANOMALY IS A SECOND-ORDER EFFECT

Let $C(\mathbf{n})$ be the speed of wave propagation, having direction $\mathbf{n}$ (at right angles to the front) in a certain asymptotic region. In that vicinity a generic *rigidly* propagating wave ψ has the form

$$\psi=\psi\left(\frac{\mathbf{n}\cdot\mathbf{R}-C(\mathbf{n})t}{\Lambda(\mathbf{n})}\right), \tag{B1}$$

where $\Lambda(\mathbf{n})$ is the cycle wavelength in direction $\mathbf{n}$, and where ψ has a unit dimensionless period:

$$\psi(z+1)=\psi(z). \tag{B2}$$

Equation (B1) gives the frequency, isotropic in the comoving system:

$$F=C(\mathbf{n})/\Lambda(\mathbf{n}). \tag{B3}$$

In the conventional case, we have $F=f_0$ (the intrinsic frequency is unaffected by the motion). In the laboratory system $\mathbf{r}=\mathbf{R}+\mathbf{V}t$ we have for the general case

$$\psi=\psi\left(\frac{\mathbf{n}\cdot\mathbf{r}}{\Lambda(\mathbf{n})}-\frac{C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}}{\Lambda(\mathbf{n})}\,t\right), \tag{B4}$$

yielding a Doppler-shifted frequency

$$F_D(\mathbf{n})=\frac{C(\mathbf{n})+\mathbf{n}\cdot\mathbf{V}}{\Lambda(\mathbf{n})}=F+\frac{\mathbf{n}\cdot\mathbf{V}}{\Lambda(\mathbf{n})}. \tag{B5}$$

This is the exact formula in terms of any existing pattern in space.

Both in the conventional and anomalous cases we have

$$\Lambda(\mathbf{n})=\lambda_0+o(V), \tag{B6}$$

and therefore to first order in V, Eq. (B5) reads

$$F_D(\mathbf{n})=F+\frac{\mathbf{n}\cdot\mathbf{V}}{\lambda_0}=f_0\left(\frac{F}{f_0}+\mathbf{n}\cdot\frac{\mathbf{V}}{c_0}\right). \tag{B7}$$

We conclude that to first order the angular pattern, measured by $F_D(\mathbf{n})-F_D(-\mathbf{n})$, is identical in the conventional and anomalous cases. It is useful to keep in mind that in the above, we have

$$\mathbf{n}=\mathbf{R}/R+o(V). \tag{B8}$$

APPENDIX C: DETAILS OF THE SIMULATIONS

The reactivity functions and parameters used in the numerical simulations are as follows. Referring to Eqs. (1) and (2), we take $\alpha=1$; the functions Φ_1, Φ_2 are

$$\Phi_1(u,v)=-\phi(u)+v, \tag{C1}$$

$$\Phi_2(u,v)=(-u+v)/\sigma(u). \tag{C2}$$

The parameter η in (2) was varied in the range (0.5, 2.75) in the unperturbed plane-wave calculations, but was set equal to 1 in simulations (a) and (b). The function ϕ is piecewise linear according to

$$\phi(u)=\begin{cases}-K_1u & \text{for } u<u_1,\\ K_2(u-a) & \text{for } u_1\leqslant u\leqslant u_2,\\ -K_3(u-1) & \text{for } u>u_2.\end{cases} \tag{C3}$$

The function σ is piecewise constant according to

$$\sigma(u)=\begin{cases}\sigma_1 & \text{for } u<B_1,\\ \sigma_2 & \text{for } B_1\leqslant u\leqslant B_2,\\ \sigma_3 & \text{for } u>B_2.\end{cases} \tag{C4}$$

The parameter values for the functions defined above, as used in this paper, are as follows: K_1=4.0, K_2=0.81 and 0.95 for simulations (a) and (b), respectively, K_3=15.0, u_1=0.018, σ_1=0.5, σ_2=16.5, σ_3=3.5, B_1=0.01, B_2=0.95. The parameters u_2 and a are determined by demanding continuity of the function $\phi(u)$: $u_2=[(K_1+K_2)u_1+K_3]/(K_3+K_2)$, $a=u_1(K_1+K_2)/K_2$. Similar parameters have been used elsewhere (e.g., Ref. [21]). For the coefficient of the linear gradient in Eq. (1) we use $G=0.16$.

Equations (1) and (2) are integrated numerically using an explicit Euler integration scheme, which has been found stable for a range of parameter values. In the work reported here, we use time and space steps of h_t=0.05 and h_x=0.8. The calculations were carried out on a SPARC 10 workstation.

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